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INTERSECTIONS OF COMPONENTS OF THE MODULI SPACE OF ABELIAN VARIETIES

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We show that over an algebraically closed field k of positive characteristic p the global moduli space of polarized abelian varieties is not the disjoint union of irreducible components. Let

$$\mathcal{A}^{(n)} = \begin{cases} \text{moduli space of pairs } (X,\lambda),\ X \text{ a} \\ g\text{-dimensional abelian variety, } \lambda : X \to X \\ \text{a polarization such that } \deg(\lambda) = n^2. \end{cases}$$

$\mathcal{A}^{(n)}$ is a scheme over Spec $\mathbf{Z}$ (for more details and references see [1]). Let $\delta = (\delta_1, \ldots, \delta_g)$ be a sequence of positive integers such that $\delta_1 | \delta_2 | \cdots | \delta_g$ and $\prod_{i=1}^{g} \delta_i = n$, and set

$$\mathcal{A}^{(\delta)} = \begin{cases} \text{the open subscheme of } \mathcal{A}^{(n)} \text{ of pairs } (X,\lambda) \\ \quad \text{such that } \ker(\lambda) \cong \prod_1^g Z/\delta_i Z \times \prod_1^g \mu_{\delta_i}. \end{cases}$$

The $\mathcal{A}^{(\delta)}$ are disjoint and their closures are the irreducible components of $\mathcal{A}^{(n)}$ [Mumford, unpublished; see [3]). The fiber of $\mathcal{A}^{(n)}$ over spec $\mathbf{C}$ is the disjoint union of the $\mathcal{A}^{(\delta)}$'s. We give here an example showing that in the fibers of $\mathcal{A}^{(n)}$ over spec(k) the closures of the $\mathcal{A}^{(\delta)}$'s are not always disjoint.

More specifically, let A be the product of a supersingular elliptic curve E with itself. Let λ_0 be a principal polarization on E and let λ be p times $\lambda_0 \times \lambda_0$. We computed the local moduli space of (A,λ) in [2] and found it has two components. We now show that one component is in the closure of $\mathcal{A}^{(p,p)}$ and the other in the closure of $\mathcal{A}^{(1,p^2)}$.

We restate the results of the computation. To do this we recall some notation. For details see [2]. Dieudonné modules over a ring R of characteristic p are certain modules over $W(R)\,[[V]]\,[F]$ where $W(R)$ is the ring of Witt vectors over R and

F, V satisfy the usual relations (including $FV = VF = p$). The Dieudonné module M of the formal group of A is given by generators e_1, e_2 and relations

$$Fe_1 = Ve_1$$

$$Fe_2 = Ve_2.$$

The dual module, M', is given by generators f_1 and f_2 and relations

$$Ff_3 = Vf_3$$

$$Ff_4 = Vf_4.$$

The universal deformation of M, written $\mathcal{M}$, is defined over the formal power series ring $k[[T_1, \ldots, T_4]]$ and is given by generators e_1, e_2 and relations

$$Fe_1 = Ve_1 + t_1e_1 + t_2e_2$$

$$Fe_2 = Ve_2 + t_3e_1 + t_4e_2.$$

Here t_i is the Witt vector $(T_i, 0, 0, \ldots)$. The dual of $\mathcal{M}$, the universal deformation of M' is given by generators f_1, f_2 and relations

$$Ff_3 = Vf_3 - t_1f_3 - t_3f_4$$

$$Ff_4 = Vf_4 - t_2f_3 - t_4f_4.$$

The polarization λ induces a map (also denoted λ)

$$\lambda : M \to M'$$

$$: e_i \mapsto p\omega f_{i+2}, \qquad i = 1, 2$$

for some unit ω in $W(k)$. The local moduli space of (A, λ) is the largest subscheme of spec $k[[T_1, \ldots, T_4]]$ on which λ extends. Setting $\mathfrak{m} = (T_1, \ldots, T_4)$ we found in [2] that this subscheme is defined by

$$(T_2 - T_3)^p (T_1(T_4 - T_2T_3) \equiv 0 \bmod \mathfrak{m}^{p+3}.$$

The extension Λ of λ is

$$(*) \qquad \begin{aligned} \Lambda &: e_1 \mapsto p\omega f_3 + \omega(t_2 - t_3)^p(-t_2f_3 - t_4f_4) \bmod \mathfrak{m}^{p+3} \\ \Lambda &: e_2 \mapsto p\omega f_4 + \omega(t_2 - t_3)^p(t_1f_3 + t_3f_4) \bmod \mathfrak{m}^{p+3}. \end{aligned}$$

The two components of the local moduli space have leading forms $T_2^p - T_3^p$ and $T_1T_4 - T_2T_3$ respectively. We assert that (1) the component defined by $(T_2 - T_3)^p$ is in the closure of $\mathcal{A}^{(p,p)}$ and (2) the component defined by $T_1T_4 - T_2T_3$ is in the closure of $\mathcal{A}^{(1,p^2)}$.

First $p^{-1} \cdot \lambda = \lambda_0 \times \lambda_0$ is a principal polarization of $A = E \times E$. An easy computation shows that $(A, p^{-1} \cdot \lambda)$ has local moduli space defined by $T_2 - T_3 +$ higher degree terms $= 0$. The first assertion follows directly from this.

The map induced by Λ from $\mathcal{M}$ to $\mathcal{M}'/V\mathcal{M}'$ is not zero on the component defined by $T_1T_4 - T_2T_3$ + higher degree terms = 0. Therefore it is not zero modulo $p = VF$. Consequently on the abelian scheme level the kernel of Λ is not contained in the kernel of multiplication by p. Thus this component is not in the closure of $\mathscr{A}^{(p,p)}$ and must be in the closure of $\mathscr{A}^{(1,p^2)}$.

References

[1] D. Mumford, The Structure of the Moduli Spaces of Curves and Abelian Varieties, Actes du Congres International des Mathematicians Nice, 1970 (Gauthier-Villars, 1971) pp. 457–466.

[2] P. Norman, An Algorithm for Computing Local Moduli of Abelian Varieties, Annals of Math. 101 (1975) 499–509.

[3] P. Norman, and F. Oort, The moduli space of abelian varieties (to appear).